

MODULAR PRINCIPAL SERIES REPRESENTATIONS

MEINOLF GECK

ABSTRACT. We present a classification of the modular principal series representations of a finite group of Lie type, in non-describing characteristic. The proofs rely on the recent progress concerning the determination of the irreducible representations of Iwahori–Hecke algebras at roots of unity.

1. INTRODUCTION

Recently, there has been considerable progress in classifying the irreducible representations of Iwahori–Hecke algebras at roots of unity. In this paper, we present an application of these results to modular Harish–Chandra series for a finite group of Lie type.

Let G be a connected reductive algebraic group defined over the finite field \mathbb{F}_q with q elements. Then $G^F = \{g \in G \mid F(g) = g\}$ is a finite group of Lie type, where $F: G \rightarrow G$ is the corresponding Frobenius map.

Let $\text{Irr}(G^F)$ be the set of complex irreducible characters of G^F . It is well-known that $\text{Irr}(G^F)$ is partitioned into Harish–Chandra series; see, for example, Carter [6, §9.2]. Let us consider the “principal series”, denoted $\text{Irr}(G^F, B^F)$ where $B \subseteq G$ is an F -stable Borel subgroup. By definition, we have $\chi \in \text{Irr}(G^F, B^F)$ if χ appears with non-zero multiplicity in the character of the permutation module $\mathbb{C}[G^F/B^F]$. Let W be the Weyl group of G , with respect to an F -stable maximal torus of B . Then F induces an automorphism of W which we denote by the same symbol. It is a classical fact (see, for example, Curtis–Reiner [7, §68B]) that we have a natural bijection

$$(\spadesuit) \quad \text{Irr}(W^F) \xrightarrow{\sim} \text{Irr}(G^F, B^F), \quad \rho \mapsto [\rho].$$

(To be more precise, the bijection depends on the choice of a square root of q ; see Lusztig [42], Geck–Pfeiffer [31].) It is the purpose of this paper to prove an analogous result for ℓ -modular Brauer characters, where ℓ is a prime not dividing q . We shall see that, if ℓ is not too small (this will be specified later), then the following hold:

- 1) the “ ℓ -modular principal series” is naturally in bijection with a *subset* of $\text{Irr}(W^F)$;
- 2) that subset is “generic”, in the sense that it only depends on

$$e = \min\{i \geq 2 \mid 1 + q + q^2 + \cdots + q^{i-1} \equiv 0 \pmod{\ell}\},$$

Date: February, 2006.

2000 *Mathematics Subject Classification.* Primary 20C33; Secondary 20C08.

but not on the particular values of q and ℓ ;

- 3) the bijection in 1) is “natural” in the sense that it fits into a more general (conjectural) bijection between unipotent characters (in the sense of Deligne–Lusztig) and unipotent Brauer characters.

If $G^F = \mathrm{GL}_n(q)$, then 1), 2) and 3) reduce to classical results due to Dipper and James [9], [10], [8]. Already for $G^F = \mathrm{GU}_n(q)$ (the general unitary group), completely new methods are needed. The results in this paper form one step towards a complete description of ℓ -modular Harish-Chandra series in general; see the articles by Hiss, Malle and the author [26], [27], [25] for a further discussion of this (as yet open) problem.

To state our results more precisely, we need to introduce some notation.

Every $\chi \in \mathrm{Irr}(G^F)$ has a well-defined *unipotent support*, denoted C_χ . This is the unique F -stable unipotent conjugacy class in G , of maximal possible dimension such that a certain average value of χ on the F -fixed points in that class is non-zero. The existence of C_χ was proved by Lusztig [44], assuming that q is a sufficiently large power of a sufficiently large prime. These conditions on q were removed by Geck–Malle [30]. We set

$$\mathbf{d}_\chi := \dim \mathfrak{B}_u \quad (u \in C_\chi)$$

where \mathfrak{B}_u is the variety of Borel subgroups containing u .

Now let ℓ be a prime not dividing q . We shall freely use the general notions of the ℓ -modular representation theory of finite groups (see Curtis–Reiner [7, §18]). Let $\mathrm{IBr}_\ell(G^F)$ be the set of irreducible Brauer characters of G^F . We have equations

$$\hat{\chi} = \sum_{\varphi \in \mathrm{IBr}_\ell(G^F)} d_{\chi, \varphi} \varphi \quad \text{for any } \chi \in \mathrm{Irr}(G^F),$$

where $\hat{\chi}$ denotes the restriction of χ to the set of ℓ -regular elements of G^F and where $d_{\chi, \varphi} \in \mathbb{Z}_{\geq 0}$ are Brauer’s ℓ -modular decomposition numbers.

Hiss [35] has shown how to define ℓ -modular Harish–Chandra series in general. In this context, the “ ℓ -modular principal series”, denoted $\mathrm{IBr}_\ell(G^F, B^F)$, is the set of all Brauer characters which are afforded by a simple quotient (or, equivalently, a simple submodule) of the permutation module $k[G^F/B^F]$, where k is a sufficiently large field of characteristic ℓ . (See also Dipper [8] for this special case.) For any $\varphi \in \mathrm{IBr}_\ell(G^F, B^F)$, we set

$$\mathbf{d}'_\varphi := \min\{\mathbf{d}_\chi \mid \chi \in \mathrm{Irr}(G^F, B^F) \text{ and } d_{\chi, \varphi} \neq 0\}.$$

We can now state the main results of this paper. We will assume throughout that G is simple modulo its center.

Theorem 1.1. *Assume that ℓ is not “too small” (see Remark 1.5 below). For any $\varphi \in \mathrm{IBr}_\ell(G^F, B^F)$, there is a unique $\rho = \rho_\varphi \in \mathrm{Irr}(W^F)$ such that*

$$d_{[\rho], \varphi} \neq 0 \quad \text{and} \quad \mathbf{d}'_\varphi = \mathbf{d}_{[\rho]}.$$

This yields an injective map $\mathrm{IBr}_\ell(G^F, B^F) \hookrightarrow \mathrm{Irr}(W^F)$, $\varphi \mapsto \rho_\varphi$.

The proof will be given in §3.A; it essentially relies on the theory of Iwahori–Hecke algebras. These come into play via the fact, due to Dipper [8, Cor. 4.10], that we have a canonical bijection between $\mathrm{IBr}_\ell(G^F, B^F)$ and the simple modules of the endomorphism algebra

$$\mathcal{E}_k := \mathrm{End}_{kG^F}(k[G^F/B^F]).$$

Using this fact, the desired injection of $\mathrm{IBr}_\ell(G^F, B^F)$ into $\mathrm{Irr}(W^F)$ follows from a suitable classification of the simple \mathcal{E}_k -modules, which is provided by the methods developed in [19], [21], [33]; see also the survey in [23].

We point out that the statements of Theorem 1.1 are false when ℓ is very small, for example, when $\ell = 2$ and G is a group of type B_n , C_n or D_n .

Theorem 1.2. *Keep the hypotheses of Theorem 1.1. Then the image of the map $\mathrm{IBr}_\ell(G^F, B^F) \hookrightarrow \mathrm{Irr}(W^F)$ only depends on e (as defined above).*

Explicit descriptions of the image of the map $\mathrm{IBr}_\ell(G^F, B^F) \hookrightarrow \mathrm{Irr}(W^F)$ are now known, through the work of several authors: Ariki [1], [3], Ariki–Mathas [4], Bremke [5], Dipper–James [9], Dipper–James–Murphy [11], Geck [18], [19], Geck–Jacon [28], Geck–Lux [29], Jacon [36], [37], [38], Müller [46]; see Theorem 3.2. Parts of these explicit results are needed in the proof of Theorem 1.2; see §3 for the details.

Finally, let us explain why we consider the parametrization in Theorem 1.1 to be “natural”. Let $\mathrm{Uch}(G^F) \subseteq \mathrm{Irr}(G^F)$ be the set of unipotent irreducible characters, as defined by Deligne–Lusztig (see [6], [43]). Let $\mathrm{UBr}_\ell(G^F)$ be the set of all $\varphi \in \mathrm{IBr}_\ell(G^F)$ such that $d_{\chi, \varphi} \neq 0$ for some $\chi \in \mathrm{Uch}(G^F)$. For $\varphi \in \mathrm{UBr}_\ell(G^F)$, we set

$$\mathbf{d}'_\varphi := \min\{\mathbf{d}_\chi \mid \chi \in \mathrm{Uch}(G^F) \text{ and } d_{\chi, \varphi} \neq 0\}.$$

Conjecture 1.3 (Geck [14, §2.5], Geck–Hiss [25, §3]). *Assume that ℓ is not “too small” and that the center of G is connected. For any $\varphi \in \mathrm{UBr}_\ell(G^F)$, there is a unique $\chi = \chi_\varphi \in \mathrm{Uch}(G^F)$ such that*

$$d_{\chi, \varphi} \neq 0 \quad \text{and} \quad \mathbf{d}'_\varphi = \mathbf{d}_\chi.$$

This yields a bijection $\mathrm{UBr}_\ell(G^F) \xrightarrow{\sim} \mathrm{Uch}(G^F)$, $\varphi \mapsto \chi_\varphi$.

Let D_{uni} be the matrix of decomposition numbers $d_{\chi, \varphi}$ where $\chi \in \mathrm{Uch}(G^F)$ and $\varphi \in \mathrm{UBr}_\ell(G^F)$. By [24], we already know that

$$|\mathrm{Uch}(G^F)| = |\mathrm{UBr}_\ell(G^F)| \quad \text{and} \quad \det D_{\mathrm{uni}} = \pm 1.$$

The above conjecture predicts that D has a block triangular shape as follows:

$$D_{\mathrm{uni}} = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ * & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & D_r \end{pmatrix}$$

where the rows are partitioned into blocks corresponding to the equivalence relation “ $\chi \sim \chi' \Leftrightarrow C_\chi = C_{\chi'}$ ”, the blocks are ordered by increasing value of \mathbf{d}_χ , and the columns are ordered via the bijection $\mathrm{UBr}_\ell(G^F) \xrightarrow{\sim} \mathrm{Uch}(G^F)$. Furthermore, each D_i is in fact an identity matrix.

The conjecture is known to be true for $G^F = \mathrm{GL}_n(q)$ (where it follows from the results of Dipper–James [10]), $G^F = \mathrm{GU}_n(q)$ (see [15]), and some explicitly worked-out examples of small rank, like $G^F = G_2(q)$ or ${}^3D_4(q)$ (see Hiss [34] and the references there).

Corollary 1.4. *Assume that ℓ is not “too small” and that Conjecture 1.3 holds for G^F . Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{IBr}_\ell(G^F, B^F) & \hookrightarrow & \mathrm{Irr}(G^F, B^F) \\ \cap & & \cap \\ \mathrm{UBr}_\ell(G^F) & \xrightarrow{\sim} & \mathrm{Uch}(G^F) \end{array}$$

where the top horizontal arrow is given by the injection in Theorem 1.1 (composed with the bijection (\spadesuit)) and the bottom horizontal arrow is given by the bijection in Conjecture 1.3.

This is an immediate consequence of the results in Section 2 (see the argument in the proof of Lemma 2.4).

Remark 1.5. We will show that the statements in Theorems 1.1 and 1.2 hold under some conditions on ℓ , which are specified as follows:

- (a) ℓ good for G^F , and $F = \text{identity on } W$; see Corollary 2.7 and §3.A.
- (b) ℓ sufficiently large (explicit bound unknown), any G, F ; see §3.A.
- (c) $G^F = \mathrm{GU}_n(q)$, any ℓ ; see §2.E. There, we also give an explicit description of the image of the map $\mathrm{Irr}(W^F) \xrightarrow{\sim} \mathrm{Irr}(G^F, B^F)$.

Here, the conditions for ℓ to be good are as follows:

$$\begin{aligned} A_n, {}^2A_n &: \text{ no condition,} \\ B_n, C_n, D_n, {}^2D_n &: \ell \neq 2, \\ G_2, {}^3D_4, F_4, {}^2E_6, E_6, E_7 &: \ell \neq 2, 3, \\ E_8 &: \ell \neq 2, 3, 5. \end{aligned}$$

We conjecture that, in general, it is sufficient to assume that ℓ is good for G^F . This is supported by the fact that the arguments for proving Theorems 1.1 and 1.2 work whenever ℓ is good and Lusztig’s conjectures on Hecke algebras with unequal parameters (as stated in [45, §14.2]) hold for W^F and the weight function given by the restriction of the length function on W to W^F . These conjectures are known to hold when F is the identity on W (the “equal parameter case”); see [45, Chap. 15]. A sketch proof for the case where F is not the identity is given in [45, Chap. 16].

We also expect that Conjecture 1.3 holds whenever ℓ is good for G^F .

Remark 1.6. As already remarked above, there is a natural bijection between $\mathrm{IBr}_\ell(G^F, B^F)$ and the set of simple modules of the algebra \mathcal{E}_k . Assume now

that W^F is of type B_n or D_n . As we shall see, in these cases, the image of the map in Theorem 1.1 is explicitly given by the results of Jacon [36], [37], [38], and Jacon and the author [28].

Note that Ariki [2] (see also Ariki–Mathas [4]) gives a different parametrization of the simple \mathcal{E}_k -modules by a subset of $\text{Irr}(W^F)$, based on the Dipper–James–Murphy theory [11] of Specht modules.

2. DECOMPOSITION NUMBERS

We keep the basic set-up of Section 1. The purpose of this section is to show how the proof of Theorem 1.1 can be reduced to an analogous statement about decomposition numbers of Iwahori–Hecke algebras. This involves three major ingredients:

- Iwahori’s realization of the endomorphism algebra \mathcal{E}_k as a specialization of a “generic” algebra; see Proposition 2.1.
- Dipper’s interpretation of the decomposition numbers $d_{\chi, \varphi}$ for $\chi \in \text{Uch}(G^F)$ and $\varphi \in \text{UBr}_\ell(G^F, B^F)$ in terms of \mathcal{E}_k ; see Proposition 2.2.
- Lusztig’s Hecke algebra interpretation for the invariants $\mathbf{d}_{[\rho]}$ where $\rho \in \text{Irr}(W^F)$; see Theorem 2.3.

Once this reduction is achieved, we can apply the results of Jacon, Rouquier and the author on the existence of so-called “canonical basic sets” for Iwahori–Hecke algebras. (For a survey, see [23]).

Let K be a sufficiently large finite field extension of \mathbb{Q} ; specifically, we require that K contains all $|G^F|$ th roots of unity and an element t such that $t^2 = q$. As is well-known, all complex irreducible characters of G^F can be realized over K , so we can actually regard $\text{Irr}(G^F)$ as the set of irreducible K -characters of G^F . A similar remark applies to $\text{Irr}(W^F)$.

Let \mathcal{O} be a discrete valuation ring in K , with residue field k of characteristic $\ell > 0$. Then it is also known that k is a splitting for G^F . Having chosen \mathcal{O} , the ℓ -modular Brauer character of a simple kG^F -module is well-defined.

For $R \in \{K, \mathcal{O}, k\}$, we define

$$\mathcal{E}_R := \text{End}_{RG^F}(R[G^F/B^F])$$

where $R[G^F/B^F]$ is the RG^F -permutation module on the cosets of B^F . By Iwahori’s theorem (see [31, §8.4] or [6, §10.10]), the algebra \mathcal{E}_R has a standard basis indexed by the elements of W^F ; furthermore, \mathcal{E}_R can be obtained from a “generic algebra”. This is done as follows.

2.A. The generic Iwahori–Hecke algebra. The group W is a finite Coxeter group with generating set S . Let $l: W \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding length function. We have $F(S) = S$. Let \bar{S} be the set of F -orbits on S . Then W^F is a finite Coxeter group with generating set $\{w_J \mid J \in \bar{S}\}$, where w_J is the longest element in the parabolic subgroup generated by J . Let $\bar{l}: W^F \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding length function. Furthermore, let $L: W^F \rightarrow \mathbb{Z}_{\geq 0}$ be the restriction of the length function on W to W^F . Then L is a weight function in the sense of Lusztig; see [45, Lemma 16.2].

Let $A = \mathcal{O}[v, v^{-1}]$ be the ring of Laurent polynomials over \mathcal{O} in an indeterminate v . Let \mathcal{H} be the generic Iwahori–Hecke algebra of W^F over A . Thus, \mathcal{H} is free as an A -module, with a basis $\{T_w \mid w \in W^F\}$, such that the following relations hold:

$$\begin{aligned} T_{ww'} &= T_w T_{w'} \quad \text{whenever } \bar{l}(ww') = \bar{l}(w) + \bar{l}(w'), \\ T_{w_J}^2 &= u^{L(w_J)} T_1 + (u^{L(w_J)} - 1) T_{w_J} \quad \text{for any } J \in \bar{S}, \text{ where } u = v^2. \end{aligned}$$

Now there is a canonical ring homomorphism $\theta_0: A \rightarrow \mathcal{O}$ such that $\theta_0(v) = t$. Similarly, there is a canonical ring homomorphism $\theta: A \rightarrow k$ such that $\theta(v) = \bar{t}$, where \bar{t} denotes the image of t in k . Thus, θ is the composition of θ_0 with the canonical map from \mathcal{O} onto k . For $R \in \{K, \mathcal{O}, k\}$, we write $\mathcal{H}_R = R \otimes_A \mathcal{H}$ where R is regarded as an A -module via the ring homomorphism θ_0 or θ .

Proposition 2.1 (Iwahori). *We have $\mathcal{H}_R \cong \mathcal{E}_R$.*

Let $K(v)$ be the field of fractions of A and write $\mathcal{H}_{K(v)} = K(v) \otimes_A \mathcal{H}$. It is known that the algebra $\mathcal{H}_{K(v)}$ is split semisimple (see Lusztig [42] for the case where L is constant on S and [31, Chap. 9] in general). By Tits' Deformation Theorem (see [31, §8.1]), the ring homomorphism $A \rightarrow K$, $v \mapsto 1$, induces a bijection between $\text{Irr}(W^F)$ and the set of simple $\mathcal{H}_{K(v)}$ -modules, up to isomorphism. Let us write

$$\text{Irr}(W^F) = \{\rho^\lambda \mid \lambda \in \Lambda\}$$

where Λ is a finite indexing set. Then, for each $\lambda \in \Lambda$, there is a well-defined simple $\mathcal{H}_{K(v)}$ -module E_v^λ . We have $\text{trace}(T_w, E_v^\lambda) \in A$ for all $w \in W^F$, and E_v^λ is uniquely determined by the condition that

$$\rho^\lambda(w) = \text{trace}(T_w, E_v^\lambda)|_{v=1} \quad \text{for all } w \in W^F.$$

Similarly, for each $\lambda \in \Lambda$, there is a corresponding simple \mathcal{H}_K -module E_t^λ , which is uniquely determined by the condition that

$$\text{trace}(T_w, E_t^\lambda) = \text{trace}(T_w, E_v^\lambda)|_{v=t} \quad \text{for all } w \in W^F.$$

In this context, the “classical” bijection (mentioned in Section 1)

$$(\spadesuit) \quad \text{Irr}(W^F) \xrightarrow{\sim} \text{Irr}(G^F, B^F), \quad \rho \mapsto [\rho],$$

is obtained as follows. Let $\rho \in \text{Irr}(W^F)$ and write $\rho = \rho^\lambda$ where $\lambda \in \Lambda$. Now regard E_t^λ as a simple \mathcal{E}_K -module via the isomorphism in Proposition 2.1. Then, by standard results on endomorphism algebras (see, for example, [31, Prop. 8.4.4] or [7, §68B]), the module E_t^λ corresponds to a well-defined irreducible character in $\text{Irr}(G^F, B^F)$, which is denoted $[\rho]$.

2.B. The decomposition matrix of \mathcal{H} . The ring homomorphism $\theta: A \rightarrow k$ induces a decomposition map

$$d_\theta: R_0(\mathcal{H}_{K(v)}) \rightarrow R_0(\mathcal{H}_k)$$

from the Grothendieck group of finitely generated $\mathcal{H}_{K(v)}$ -modules to the Grothendieck group of finitely generated \mathcal{H}_k -modules (see [20, §2] or [23, §4]). Given a simple $\mathcal{H}_{K(v)}$ -module E and a simple \mathcal{H}_k -module M , we denote

$$(E : M) = \begin{array}{l} \text{multiplicity of } M \text{ in the image of } E \\ \text{(under the decomposition map } d_\theta\text{).} \end{array}$$

Finally, let us write the set of simple \mathcal{H}_k -modules (up to isomorphism) as

$$\{M^\mu \mid \mu \in \Lambda^\circ\} \quad \text{where } \Lambda^\circ \text{ is a finite indexing set.}$$

Using Dipper's Hom functors [8], the set Λ° also canonically parametrizes the ℓ -modular principal series of G^F :

$$\text{IBr}_\ell(G^F, B^F) = \{\varphi^\mu \mid \mu \in \Lambda^\circ\}.$$

With these notations, we can now state the following basic result.

Proposition 2.2 (Dipper). *Let $\chi \in \text{Uch}(G^F)$ and $\mu \in \Lambda^\circ$. Then*

$$d_{\chi, \varphi^\mu} = \begin{cases} (E_v^\lambda : M^\mu) & \text{if } \chi = [\rho^\lambda] \text{ where } \lambda \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, Dipper's original result in [8, Corollary 4.10] works with the decomposition numbers of the algebra \mathcal{E}_O . The above formulation takes into account the factorization result in [20, 3.3], in order to lift the statement to the generic algebra \mathcal{H} .

2.C. Lusztig's \mathbf{a} -function. The algebra \mathcal{H} is symmetric, with symmetrizing trace $\tau: \mathcal{H} \rightarrow A$ given by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W^F$. Since $\mathcal{H}_{K(v)}$ is split semisimple and A is integrally closed in $K(v)$, there are well-defined Laurent polynomials $\mathbf{c}_\lambda \in A$ such that

$$\tau(T_w) = \sum_{\lambda \in \Lambda} \frac{1}{\mathbf{c}_\lambda} \text{trace}(T_w, E_v^\lambda) \quad \text{for all } w \in W^F.$$

This follows from a general argument concerning symmetric algebras; see [31, Chapter 7]. Since the weight function L arises from a finite group of Lie type, we actually have $\mathbf{c}_\lambda \in \mathbb{Z}[u, u^{-1}]$ and

$$[\rho^\lambda](1) = \frac{[G^F : B^F]}{\mathbf{c}_\lambda(q)} \quad \text{for all } \lambda \in \Lambda;$$

see Curtis–Reiner [7, §68C]. Let us write

$$\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} + \text{combination of higher powers of } u,$$

where $f_\lambda \neq 0$ and $\mathbf{a}_\lambda \geq 0$ are integers. This yields Lusztig's \mathbf{a} -function

$$\text{Irr}(W^F) \rightarrow \mathbb{Z}_{\geq 0}, \quad \rho^\lambda \mapsto \mathbf{a}_\lambda.$$

This function is related to the \mathbf{d} -invariants considered in Section 1 by the following result.

Theorem 2.3 (Lusztig, Geck–Malle). *Let $\lambda \in \Lambda$ and C be the unipotent support of $[\rho^\lambda] \in \text{Irr}(G^F, B^F)$. Then we have*

$$\mathbf{a}_\lambda = \mathbf{d}_{[\rho^\lambda]} = \dim \mathfrak{B}_u \quad (u \in C).$$

This was proved by Lusztig [44, §10], assuming that q is a power of a sufficiently large prime. These conditions were removed by [30, Prop. 3.6].

2.D. Canonical basic sets. Using the \mathbf{a} -function, we now define the notion of a “canonical basic set”, following [23, Def. 4.13]. For any $\mu \in \Lambda^\circ$, we set

$$\mathbf{a}'_\mu = \min\{\mathbf{a}_\lambda \mid \lambda \in \Lambda \text{ and } (E_v^\lambda : M^\mu) \neq 0\}.$$

Let $\iota: \Lambda^\circ \rightarrow \Lambda$ be an injective map and write $\mathcal{B}_{k,q} = \iota(\Lambda^\circ) \subseteq \Lambda$. We say that $\mathcal{B}_{k,q}$ is a *canonical basic set* for \mathcal{H}_k if the following conditions are satisfied.

$$\begin{aligned} (E_v^{\iota(\mu)} : M^\mu) &= 1 & \text{for all } \mu \in \Lambda^\circ, \\ (E_v^\lambda : M^\mu) &= 0 & \text{unless } \mathbf{a}'_\mu < \mathbf{a}_\lambda \text{ or } \lambda = \iota(\mu). \end{aligned}$$

Note that, if it exists, then ι is uniquely determined by these conditions: given $\mu \in \Lambda^\circ$, the image $\iota(\mu)$ is the unique $\lambda \in \Lambda$ such that

$$(E_v^\lambda : M^\mu) \neq 0 \quad \text{and} \quad \mathbf{a}'_\mu = \mathbf{a}_\lambda.$$

The following result provides the desired reduction of Theorem 1.1 to a statement entirely in the framework of Iwahori–Hecke algebras.

Lemma 2.4. *Assume that \mathcal{H}_k admits a canonical basic set $\mathcal{B}_{k,q} = \iota(\Lambda^\circ)$, as above. Then the statements in Theorem 1.1 hold for G^F , where*

$$\rho_{\varphi^\mu} = \rho^{\iota(\mu)} \quad \text{for all } \mu \in \Lambda^\circ.$$

Furthermore, assuming that Conjecture 1.3 holds, we have a commutative diagram as in Corollary 1.4.

Proof. This is an immediate consequence of Proposition 2.2, taking into account the identities in Theorem 2.3. \square

Now we have the following general existence result.

Theorem 2.5 (Geck [19], [21], [23] and Geck–Rouquier [33]). *Assume that Lusztig’s conjectures (P1)–(P14) in [45, §14.2] and a certain weak version of (P15) (as specified in [23, 5.2]) hold for \mathcal{H} . Assume further that ℓ is good for G^F . Then \mathcal{H}_k admits a canonical basic set.*

Remark 2.6. (a) The above result is proved by a general argument, using deep properties of the Kazhdan–Lusztig basis of \mathcal{H} . These properties are known to hold, for example, when F is the identity on W (the “equal parameter case”); see Lusztig [45, Chap. 15]. A sketch proof for the case where F is not the identity is given by Lusztig [45, Chap. 16].

(b) If ℓ is not good, it is easy to produce examples in which a canonical basic set does not exist; see [23, 4.15].

Corollary 2.7. *Assume that ℓ is a good prime for G^F and F is the identity on W . Then the statements in Theorem 1.1 and Corollary 1.4 hold for G^F .*

Proof. By Lusztig [45, Chap. 15], the hypotheses of Theorem 2.5 are satisfied and so \mathcal{H}_k admits a canonical basic set. It remains to use Lemma 2.4. \square

2.E. The finite unitary groups. Let $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and F be such that $G^F = \mathrm{GU}_n(q)$, the finite general unitary group. Then W^F is a Coxeter group of type B_m , where $m = n/2$ (if n is even) or $m = (n-1)/2$ (if n is odd). Writing $n = 2m + s$, the weight function $L: W^F \rightarrow \mathbb{Z}_{\geq 0}$ is given by

$$B_m \quad \begin{array}{ccccccc} & 2s+1 & 2 & 2 & & & 2 \\ & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \end{array}$$

All unipotent classes in G are F -stable, and they are naturally labelled by the partitions of n . Given $\lambda \vdash n$, let C_λ be the class containing matrices of Jordan type λ . For $u \in C_\lambda$, we have

$$\dim \mathfrak{B}_u = n(\lambda) := \sum_{i=1}^r (i-1)\lambda_i$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0)$. The unipotent characters are also labelled by the partitions of n . Thus, we can write

$$\mathrm{Uch}(G^F) = \{\chi^\lambda \mid \lambda \vdash n\}.$$

For example, $\chi^{(n)}$ is the unit and $\chi^{(1^n)}$ is the Steinberg character. We have

$$C_{\chi^\lambda} = C_\lambda \quad \text{for all } \lambda \vdash n.$$

See Lusztig [41, §9], Kawanaka [40] and Carter [6, Chap. 13] for further details. Now let ℓ be any prime not dividing q . By [15], we have a parametrization $\mathrm{UBr}_\ell(G^F) = \{\varphi^\mu \mid \mu \vdash n\}$ such that the following hold:

$$\begin{aligned} d_{\chi^\lambda, \varphi^\lambda} &= 1 & \text{for all } \lambda \vdash n, \\ d_{\chi^\lambda, \varphi^\mu} &= 0 & \text{unless } \lambda \leq \mu, \end{aligned}$$

where \leq denotes the dominance order on partitions. (The proof essentially relies on Kawanaka's theory [40] of generalized Gelfand–Graev representations.)

Now, it is known that, for any $\nu, \nu' \vdash n$, we have $\nu \leq \nu' \Rightarrow n(\nu') \leq n(\nu)$, with equality only if $\nu = \nu'$ (see, for example, [31, Exc. 5.6]). Hence the above conditions on the decomposition numbers can also be phrased as:

$$\begin{aligned} d_{\chi^\lambda, \varphi^\lambda} &= 1 & \text{for all } \lambda \vdash n, \\ d_{\chi^\lambda, \varphi^\mu} &= 0 & \text{unless } n(\mu) < n(\lambda) \text{ or } \lambda = \mu. \end{aligned}$$

Using the formula for $\dim \mathfrak{B}_u$, it is now clear that Conjecture 1.3 holds.

Now let us consider the principal series characters. The set $\mathrm{Irr}(W^F)$ is naturally parametrized by the set Λ of pairs of partitions of total size m . The inclusion $\mathrm{Irr}(G^F, B^F) \subseteq \mathrm{Uch}(G^F)$ corresponds to an embedding of Λ into the set of partitions of n , which is explicitly described in the appendix

of [13], based on Lusztig [41, §9]. (The description involves the notions of the 2-core and the 2-quotient of a partition.)

Using Proposition 2.2 and the identity in Theorem 2.3, we conclude that the statements in Theorem 1.1 and Corollary 1.4 also hold for G^F . Note, however, that these are pure existence results! An explicit combinatorial description of the image of the map $\text{IBr}_\ell(G^F, B^F) \hookrightarrow \text{Irr}(W^F)$ (or, equivalently, of a canonical basic set for \mathcal{H}_k) is much harder to obtain. The complete answer was only achieved quite recently; see [28].

If $e = 2$, then that image is given by a class of bipartitions described in [28, Theorem 3.4]; if $e > 2$ is twice an odd number, then that image is given by a class of bipartitions defined by Foda et al. [12]; see [28, Theorem 5.4]; otherwise, that image is given by the set of all pairs of e -regular partitions of total size m ; see [28, Theorem 3.1]. (These results hold for any prime ℓ not dividing q .)

3. INDEPENDENCE OF CANONICAL BASIC SETS

In order to establish the independence statement in Theorem 1.2, we use a technique originally developed in [16, §4] (see also [32, §2] and [20, §2]), namely, a factorization of the decomposition map

$$d_\theta: R_0(\mathcal{H}_{K(v)}) \rightarrow R_0(\mathcal{H}_k).$$

Let $e = \min\{j \geq 2 \mid 1 + q + q^2 + \cdots + q^{j-1} \equiv 0 \pmod{\ell}\}$ as before and $\zeta_e := \exp(2\pi i/e) \in \mathbb{C}$. Choosing \mathcal{O} suitably, we may assume that \mathcal{O} contains an element ζ'_e such that $\zeta_e'^2 = \zeta_e$ and t, ζ'_e have the same image in k . Then we have a canonical ring homomorphism $\theta_e: A \rightarrow \mathcal{O}$ such that $\theta_e(v) = \zeta'_e$, and θ is the composition of θ_e and the canonical map $\mathcal{O} \rightarrow k$. We consider the Iwahori–Hecke algebra

$$\mathcal{H}_{\mathcal{O}}^{(e)} = \mathcal{O} \otimes_A \mathcal{H}$$

where \mathcal{O} is considered as an A -module via θ_e . Thus, in $\mathcal{H}_{\mathcal{O}}^{(e)}$, we have

$$T_{w_J}^2 = \zeta_e^{L(w_J)} T_1 + (\zeta_e^{L(w_J)} - 1) T_{w_J} \quad \text{for any } J \in \bar{S}.$$

The map $\theta_e: A \rightarrow \mathcal{O}$ induces a well-defined decomposition map

$$d_e: R_0(\mathcal{H}_{K(v)}) \rightarrow R_0(\mathcal{H}_K^{(e)})$$

between the Grothendieck groups of $\mathcal{H}_{K(v)}$ and $\mathcal{H}_K^{(e)} = K \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}^{(e)}$. Similarly, the canonical map $\mathcal{O} \rightarrow k$ induces a decomposition map

$$d': R_0(\mathcal{H}_K^{(e)}) \rightarrow R_0(\mathcal{H}_k).$$

By [20, 2.6], we then have the following factorization of d_θ :

$$\begin{array}{ccc} R_0(\mathcal{H}_{K(v)}) & \xrightarrow{d_\theta} & R_0(\mathcal{H}_k) \\ & \searrow d_e \quad \nearrow d' & \\ & R_0(\mathcal{H}_K^{(e)}) & \end{array}$$

Using this factorisation, we obtain the following result which provides a plan for proving Theorem 1.2.

Lemma 3.1 (Jacon [36, Théorème 3.1.3]). *Assume that the following hold:*

- (a) \mathcal{H}_k admits a canonical basic set, say $\mathcal{B}_{k,q}$.
- (b) $\mathcal{H}_K^{(e)}$ admits a canonical basic set, say \mathcal{B}_e .
- (c) \mathcal{H}_k and $\mathcal{H}_K^{(e)}$ have the same number of simple modules (up to isomorphism).

Then we have $\mathcal{B}_e = \mathcal{B}_{k,q}$. In particular, $\mathcal{B}_{k,q}$ only depends on e .

Proof. In order to illustrate the use of the above factorization, we give the argument here. Recall our notation $\{E_v^\lambda \mid \lambda \in \Lambda\}$ and $\{M^\mu \mid \mu \in \Lambda^\circ\}$ for the simple modules of $\mathcal{H}_{K(v)}$ and \mathcal{H}_k , respectively.

By (c), we have a labelling of the simple $\mathcal{H}_K^{(e)}$ -modules

$$\{M_e^\mu \mid \mu \in \Lambda_e^\circ\} \quad \text{where} \quad |\Lambda_e^\circ| = |\Lambda^\circ|.$$

The above factorization means that we have the following identity:

$$(E_v^\lambda : M^\mu) = \sum_{\nu \in \Lambda_e^\circ} (E_v^\lambda : M_e^\nu) \cdot (M_e^\nu : M^\mu)$$

for all $\lambda \in \Lambda$ and $\mu \in \Lambda^\circ$, where $(E_v^\lambda : M_e^\nu)$ are the decomposition numbers of $\mathcal{H}_K^{(e)}$. By (a), we have a canonical basic set $\mathcal{B}_{k,q} = \iota(\Lambda^\circ)$ for \mathcal{H}_k , where $\iota : \Lambda^\circ \rightarrow \Lambda$ is an injection. By (b), we have a canonical basic set $\mathcal{B}_e = \iota_e(\Lambda_e^\circ)$ for $\mathcal{H}_K^{(e)}$, where $\iota_e : \Lambda_e^\circ \rightarrow \Lambda$ is an injection.

We now define a map $\beta : \Lambda^\circ \rightarrow \Lambda_e^\circ$, as follows. Let $\mu \in \Lambda^\circ$ and $\lambda = \iota(\mu)$. Then $(E_v^\lambda : M^\mu) = 1$ and so there is a unique $\mu_0 \in \Lambda_e^\circ$ such that $(E_v^\lambda : M_e^{\mu_0}) \neq 0$ and $(M_e^{\mu_0} : M^\mu) \neq 0$; we set $\beta(\mu) = \mu_0$. Now we claim that

$$(*) \quad \iota(\mu) = \iota_e(\beta(\mu)) \quad \text{for all } \mu \in \Lambda^\circ.$$

This is seen as follows. Let $\mu \in \Lambda^\circ$ and $\lambda = \iota(\mu)$. By the construction of $\mu_0 = \beta(\mu)$, we have $(E_v^\lambda : M_e^{\mu_0}) = 1$ and $(M_e^{\mu_0} : M^\mu) = 1$. It remains to show that $\mathbf{a}_{\lambda'} > \mathbf{a}_\lambda$ for any $\lambda' \in \Lambda$ such that $\lambda \neq \lambda'$ and $(E_v^{\lambda'} : M_e^{\mu_0}) \neq 0$. Indeed, we have

$$\begin{aligned} (E_v^{\lambda'} : M^\mu) &= \sum_{\nu \in \Lambda_e^\circ} (E_v^{\lambda'} : M_e^\nu) \cdot (M_e^\nu : M^\mu) \\ &= (E_v^{\lambda'} : M_e^{\mu_0}) \cdot (M_e^{\mu_0} : M^\mu) + \text{further terms,} \end{aligned}$$

and so this decomposition number is non-zero. Since $\mathcal{B}_{k,q}$ is a canonical basic set, we must have $\mathbf{a}_{\lambda'} > \mathbf{a}_\lambda$, as desired. Thus, $(*)$ is proved. This relation implies that $\mathcal{B}_{k,q} \subseteq \mathcal{B}_e$. Since $\mathcal{B}_{k,q}$ and \mathcal{B}_e have the same cardinality, we conclude that $\mathcal{B}_{k,q} = \mathcal{B}_e$, as desired. \square

We are now going to show that the conditions in Lemma 3.1 are satisfied.

Theorem 3.2. *The algebra $\mathcal{H}_K^{(e)}$ admits a canonical basic set. This canonical basic set is explicitly known in all cases.*

Proof. This is proved by a combination of various results. If F acts as the identity on W , then the existence of a canonical basic is covered by Theorem 2.5 (which also works for K instead of k). For type A_{n-1} , an explicit description is given in [19, Exp. 3.5] (based on Dipper–James [9]); for type B_n (with equal parameters) and type D_n , see Jacon [36], [37], [38]; for the exceptional types G_2 , F_4 , E_6 , E_7 , E_8 (with equal parameters), see the tables of Jacon [36, §3.3]. These tables are derived from the results on decomposition numbers by Geck–Lux [29], Geck [17], [18], and Müller [46].

Assume now that F is not the identity on W . Then W^F is of type G_2 (with parameters q^3, q), F_4 (with parameters q^2, q^2, q, q) or B_m . In the first two cases, the existence of a canonical basic set follows from the explicit determination of the decomposition matrices by [14, Satz 3.13.1] (type G_2) and Bremke [5] (type F_4). The \mathbf{a} -function for type F_4 (with possibly unequal parameters) is printed in [23, Exp. 4.8].

For the convenience of the reader, let us give the details for type G_2 with parameters q^3, q . Denote the two Coxeter generators of W^F by α and β , where $L(\alpha) = 3$ and $L(\beta) = 1$. By [31, 8.3.1], the algebra $\mathcal{H}_{K(v)}$ has four 1-dimensional irreducible representations:

$$\begin{aligned} \text{ind}: \quad T_\alpha &\mapsto u^3, & T_\beta &\mapsto u; \\ \varepsilon: \quad T_\alpha &\mapsto -1, & T_\beta &\mapsto -1; \\ \varepsilon_1: \quad T_\alpha &\mapsto u^3, & T_\beta &\mapsto -1; \\ \varepsilon_2: \quad T_\alpha &\mapsto -1, & T_\beta &\mapsto u; \end{aligned}$$

and two 2-dimensional representations:

$$\rho_\delta: T_\alpha \mapsto \begin{pmatrix} -1 & 0 \\ u^2 + \delta u + 1 & u^3 \end{pmatrix}, \quad T_\beta \mapsto \begin{pmatrix} u & u \\ 0 & -1 \end{pmatrix},$$

where $\delta = \pm 1$. The \mathbf{a} -invariants are obtained from [31, 8.3.4]. The decomposition numbers are given as follows:

ρ_λ	\mathbf{a}_λ	$e = 2$	$e = 3$	$e = 6$	$e = 12$
ind	0	1 . .	1 . . .	1 . .	1
ε_1	1	1 . .	. 1 . .	. 1 .	. 1 . . .
ρ_+	3	. 1 .	. 1 1 .	. . 1	1 . 1 . .
ρ_-	3	. . 1	1 . . 1	1 1 1 .
ε_2	7	1 1 .	1 1
ε	12	1 1	. 1 .	. . 1 . .

This yields the following canonical basic sets:

$$\begin{aligned} e = 2: & \quad \{\text{ind}, \rho_+, \rho_-\}, \\ e = 3: & \quad \{\text{ind}, \varepsilon_1, \rho_+, \rho_-\}, \\ e = 6: & \quad \{\text{ind}, \varepsilon_1, \rho_+\}, \\ e = 12: & \quad \{\text{ind}, \varepsilon_1, \rho_+, \rho_-, \varepsilon_2\}. \end{aligned}$$

For all other values of e , the decomposition matrix is the identity matrix. Thus, in each case, we see that a canonical basic sets exists.

Finally, the existence of a canonical basic set in type B_m (for any choice of the parameters) is established by Geck–Jacon [28]. \square

In those cases where the hypotheses of Theorem 2.5 (concerning Lusztig’s conjectures on Hecke algebras with unequal parameters) are not yet known to hold, we now have the following result:

Corollary 3.3. *There is a constant N , depending only on W , such that \mathcal{H}_k admits a canonical basic set, whenever $\ell > N$.*

Proof. By [16, Prop. 5.5] (see also [20, §2.7]), there is a constant N , depending only on W , such that d' is an isomorphism preserving the classes of simple modules, whenever $\ell > N$. Thus, \mathcal{H}_k and $\mathcal{H}_K^{(e)}$ have the same decomposition numbers, whenever $\ell > N$. The assertion then follows from the existence of a canonical basic set for $\mathcal{H}_K^{(e)}$, by Theorem 3.2. \square

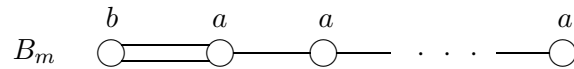
Theorem 3.4. *Assume that ℓ is good for G^F if F is the identity on W , and ℓ does not divide the order of W^F , otherwise. Then $\mathcal{H}_K^{(e)}$ and \mathcal{H}_k have the same number of simple modules (up to isomorphism). In particular, this number only depends on e .*

Note that, even if \mathcal{H}_k and $\mathcal{H}_K^{(e)}$ have the same number of simple modules, d' may still have a non-trivial decomposition matrix. For examples, see the “adjustment matrices” computed by James [39].

Proof. In [32], it is shown by a general argument that the result is true if ℓ does not divide the order of W . Hence, for each type of W , there is only a finite number of additional cases to be considered.

If W is of exceptional type, these additional cases can be handled by explicit computations with the character tables of $\mathcal{H}_{K(v)}$; in the case where F is the identity on W , this has been worked out explicitly in [22]. Similar methods apply to the cases where W is of type D_4 or E_6 , and W^F is of type G_2 or F_4 , respectively. The desired equality can also be seen directly from the decomposition numbers computed in [14, Satz 3.13.1] (for type G_2 with parameters q^3, q) and Bremke [5] (for type F_4 with parameters q^2, q^2, q, q). In both of these cases, it is sufficient to assume that $\ell \neq 2, 3$.

It remains to consider W of type A_{n-1} , B_n or D_n . In type A_{n-1} , the result is known by Dipper–James [9] (the required number is the number of e -regular partitions of n). For type D_n , we have a reduction to type B_n by [21, Theorem 6.3]. Thus, finally, we have to deal with the case where F acts on W (of type A_{n-1} , B_n or D_n) such that W^F is of type B_m (where m depends on n and F). The weight function L on W^F is specified by two integers $a, b \geq 0$ such that:



We have the following possibilities:

- (i) $a = b = 1$, where G^F is of type B_n or C_n (and $m = n$).
- (ii) $a = 2$, $b = 2s + 1$, where G^F is of type ${}^2A_{n-1}$ (and $n = 2m + s$, $s \in \{0, 1\}$).
- (iii) $a = 1$, $b = 2$, where G^F is of type 2D_n (and $n = m + 1$).
- (iv) $a = 1$, $b = 0$, where G^F is of type D_n (and $n = m$).

If $q^a \equiv 1 \pmod{\ell}$, the assertion follows from Dipper–James–Murphy [11, Theorem 7.3]. (Note that, in case (ii), we assume that ℓ does not divide the order of W^F .) Assume from now on that $q^a \not\equiv 1 \pmod{\ell}$. Then, by Ariki–Mathas [4, Theorem A], we have:

- (*) *The number of simple \mathcal{H}_k -modules (up to isomorphism) only depends on e' and A ,*

where

$$e' := \min\{j \geq 2 \mid 1 + q^a + q^{2a} + \cdots + q^{a(j-1)} \equiv 0 \pmod{\ell}\},$$

$$A := \{j \in \mathbb{Z} \mid \theta(q^b + q^{aj}) = 0\}.$$

Similarly, the number of simple $\mathcal{H}_K^{(e)}$ -modules only depends on e' and $A_0 := \{j \in \mathbb{Z} \mid \zeta_e^b + \zeta_e^{aj} = 0\}$. Hence, all we need to do is to check that $A = A_0$. Now, since $q^a \not\equiv 1 \pmod{\ell}$, we have $e = e'$ unless $a = 2$ and e is even, in which case $e' = e/2$. Furthermore, e is the multiplicative order of q modulo ℓ . In particular, e is coprime to ℓ . Using the fact that the canonical map $\mathcal{O} \rightarrow k$ induces an isomorphism on roots of unity of order prime to ℓ , it readily follows that $A = A_0$ in each of the above cases, as required. \square

3.A. Proofs of Theorem 1.1 and 1.2. By Theorem 2.5 and Corollary 3.3, the algebra \mathcal{H}_k admits a canonical basic set, say $\mathcal{B}_{k,q}$, if one of the following holds:

- ℓ is good for G^F and F acts as the identity on W ;
- ℓ is sufficiently large (where the bound comes from the proof of Corollary 3.3).

In these cases, Theorem 1.1 and Corollary 1.4 hold for G^F by Lemma 2.4.

As far as Theorem 1.2 is concerned, we first note that Lemma 2.4 provides a reduction to an analogous statement for $\mathcal{B}_{k,q}$. Thus, it remains to show that $\mathcal{B}_{k,q}$ only depends on e . But this follows from Theorem 3.4, Theorem 3.2 and Lemma 3.1.

REFERENCES

- [1] S. ARIKI, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, J. Math. Kyoto Univ. **36** (1996), 789–808.
- [2] S. ARIKI, On the classification of simple modules for cyclotomic Hecke algebras of type $G(m, 1, n)$ and Kleshchev multipartitions, Osaka J. Math. **38** (2001), 827–837.
- [3] S. ARIKI, Representations of quantum algebras and combinatorics of Young tableaux, University Lecture Series **26**, Amer. Math. Soc., Providence, RI, 2002.
- [4] S. ARIKI AND A. MATHAS, The number of simple modules of the Hecke algebras of type $G(r, 1, n)$, Math. Z. **233** (2000), 601–623.

- [5] K. BREMKE, The decomposition numbers of Hecke algebras of type F_4 with unequal parameters, *Manuscripta Math.* **83** (1994), 331–346.
- [6] R. W. CARTER, Finite groups of Lie type: Conjugacy classes and complex characters, Wiley, New York, 1985.
- [7] C. W. CURTIS AND I. REINER, Methods of representation theory Vol. I and II, Wiley, New York, 1981 and 1987.
- [8] R. DIPPER, On quotients of Hom-functors and representations of finite general linear groups I, *J. Algebra* **130** (1990), 235–259; II, *J. Algebra* **209** (1998), 199–269.
- [9] R. DIPPER AND G. D. JAMES, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* **52** (1986), 20–52.
- [10] R. DIPPER AND G. D. JAMES, The q -Schur algebra, *Proc. London Math. Soc.* **59** (1989), 23–50.
- [11] R. DIPPER, G. D. JAMES AND G. E. MURPHY, Hecke algebras of type B_n at roots of unity, *Proc. London Math. Soc.* **70** (1995), 505–528.
- [12] O. FODA, B. LECLERC, M. OKADO, J.-Y. THIBON AND T. WELSH, Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras, *Advances in Math.* **141** (1999), 322–365.
- [13] P. FONG AND B. SRINIVASAN, Brauer trees in classical groups, *J. Algebra* **131** (1990), 179–225.
- [14] M. GECK, Verallgemeinerte Gelfand-Graev Charaktere und Zerlegungszahlen endlicher Gruppen vom Lie-Typ, Dissertation, RWTH Aachen, 1990.
- [15] M. GECK, On the decomposition numbers of the finite unitary groups in non-defining characteristic, *Math. Z.* **207** (1991), 83–89.
- [16] M. GECK, Brauer trees of Hecke algebras, *Comm. Algebra* **20** (1992), 2937–2973.
- [17] M. GECK, The decomposition numbers of the Hecke algebra of type E_6 , *Math. Comp.* **61** (1993), 889–899.
- [18] M. GECK, Beiträge zur Darstellungstheorie von Iwahori–Hecke–Algebren, Habilitationsschrift, Aachener Beiträge zur Mathematik **11**, Verlag der Augustinus Buchhandlung, Aachen, 1995.
- [19] M. GECK, Kazhdan-Lusztig cells and decomposition numbers. *Represent. Theory* **2** (1998), 264–277 (electronic).
- [20] M. GECK, Representations of Hecke algebras at roots of unity, *Séminaire Bourbaki*, année 1997-98, Astérisque No. 252 (1998), Exp. 836, 33–55.
- [21] M. GECK, On the representation theory of Iwahori-Hecke algebras of extended finite Weyl groups. *Represent. Theory* **4** (2000), 370–397 (electronic).
- [22] M. GECK, On the number of simple modules of Iwahori–Hecke algebras of finite Weyl groups. *Bul. Stiit. Univ. Baia Mare, Ser. B* **16** (2000), 235–246; preprint available at <http://arXiv.org/math.RT/0405555>
- [23] M. GECK, Modular representations of Hecke algebras. *In: Group representation theory*, Presses Polytechniques et Universitaires Romandes, EPFL-Press, 2006 (to appear).
- [24] M. GECK AND G. HISS, Basic sets of Brauer characters of finite groups of Lie type, *J. reine angew. Math.* **418** (1991), 173–188.
- [25] M. GECK AND G. HISS, Modular representations of finite groups of Lie type in non-defining characteristic, in: *Finite reductive groups: Related structures and representations* (ed. M. Cabanes), pp. 195–249. Birkhäuser, Basel, 1997.
- [26] M. GECK, G. HISS, AND G. MALLE, Cuspidal unipotent Brauer characters, *J. Algebra* **168** (1994), 182–220.
- [27] M. GECK, G. HISS, AND G. MALLE, Towards a classification of the irreducible representations in non-defining characteristic of a finite group of Lie type, *Math. Z.* **221** (1996), 353–386.
- [28] M. GECK AND N. JACON, Canonical basic sets in type B , *J. Algebra* (to appear).

- [29] M. GECK AND K. LUX, The decomposition numbers of the Hecke algebra of type F_4 . *Manuscripta Math.* **70** (1991), 285–306.
- [30] M. GECK AND G. MALLE, On the existence of a unipotent support for the irreducible characters of finite groups of Lie type. *Trans. Amer. Math. Soc.* **352** (2000), 429–456.
- [31] M. GECK AND G. PFEIFFER, Characters of finite Coxeter groups and Iwahori–Hecke algebras, *London Math. Soc. Monographs, New Series* **21**, Oxford University Press, New York 2000. xvi+446 pp.
- [32] M. GECK AND R. ROUQUIER, Centers and simple modules for Iwahori–Hecke algebras. *In: Finite reductive groups: Related structures and representations* (ed. M. Cabanes), pp. 251–272. Birkhäuser, Basel, 1997.
- [33] M. GECK AND R. ROUQUIER, Filtrations on projective modules for Iwahori–Hecke algebras. *In: Modular Representation Theory of Finite Groups* (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 211–221, Walter de Gruyter, Berlin 2001.
- [34] G. HISS, Decomposition numbers of finite groups of Lie type in non-defining characteristic, *in: G. O. Michler and C. M. Ringel, Eds., Representation Theory of Finite Groups and Finite-Dimensional Algebras*, Birkhäuser, 1991, pp. 405–418.
- [35] G. HISS, Harish-Chandra series of Brauer characters in a finite group with a split BN -pair, *J. London Math. Soc.* **48** (1993), 219–228.
- [36] N. JACON, Représentations modulaires des algèbres de Hecke et des algèbres de Ariki-Koike, Thèse de Doctorat, Université Lyon 1, 2004; available at “theses-ON-line” <http://tel.ccsd.cnrs.fr/documents/archives0/00/00/63/83>.
- [37] N. JACON, Sur les représentations modulaires des algèbres de Hecke de type D_n , *J. Algebra* **274** (2004), 607–628.
- [38] N. JACON, On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity, *J. Math. Kyoto Univ.* **44** (2004), 729–767.
- [39] G. D. JAMES, The decomposition matrices of $GL_n(q)$ for $n \leq 10$, *Proc. London Math. Soc.* **60** (1990), 225–265.
- [40] N. KAWANAKA, Generalized Gelfand-Graev representations and Ennola Duality. *In: Algebraic Groups and Related Topics*, *Advanced Studies in Pure Math.* **6**, Kinokuniya, Tokyo, and North-Holland, Amsterdam, 1985, pp. 175–206.
- [41] G. LUSZTIG, Irreducible representations of finite classical groups, *Invent. Math.* **43** (1977), 125–175.
- [42] G. LUSZTIG, On a theorem of Benson and Curtis, *J. Algebra* **71** (1981), 490–498.
- [43] G. LUSZTIG, Characters of reductive groups over a finite field, *Annals Math. Studies*, vol. 107, Princeton University Press, 1984.
- [44] G. LUSZTIG, A unipotent support for irreducible representations, *Adv. Math.* **94** (1992), 139–179.
- [45] G. LUSZTIG, Hecke algebras with unequal parameters, *CRM Monographs Ser.* **18**, Amer. Math. Soc., Providence, RI, 2003.
- [46] J. MÜLLER, Zerlegungszahlen für generische Iwahori–Hecke-Algebren von exzeptionellem Typ, Dissertation, RWTH Aachen, 1995.

DEPARTMENT OF MATHEMATICAL SCIENCES, KING’S COLLEGE, ABERDEEN UNIVERSITY, ABERDEEN AB24 3UE, SCOTLAND, U.K.

E-mail address: `m.geck@maths.abdn.ac.uk`